

Computation of Controllable and Coobservable Sublanguages in Decentralized Supervisory Control via Communication

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Abstract

In decentralized supervisory control, several local supervisors cooperate to accomplish a common goal (specification). Controllability and coobservability are the key conditions to achieve a specification in the controlled system. We construct a controllable and coobservable sublanguage of the specification by using additional communications between supervisors. Namely, we extend the observable events of local supervisors via communication and apply a fully decentralized computation of local supervisors. Coobservability is then guaranteed by construction. We also discuss sufficient conditions to achieve the centralized optimal solution. Our approach can be used for both prefix-closed and non-prefix-closed specifications.

Index terms— Discrete-event systems Decentralized supervisory control Coobservability Separability Communication

1 Introduction

Supervisory control theory of discrete-event systems modeled by finite automata was introduced by Ramadge and Wonham [31]. It aims to guarantee that the control specification consisting of safety and/or nonblockingness is satisfied in the controlled system. Supervisory control is realized by a supervisor that runs in parallel with the system and imposes the specification by disabling some of the controllable events in a feedback manner.

Decentralized supervisory control was developed by Rudie and Wonham [39]. It is based on the idea to distribute the actuator and sensor capabilities among several local supervisors. Each supervisor issues a control decision based on its own observation of the system. The global control action is then given by a fusion rule on the local control actions.

To give an example, consider a traffic system of a region with many elements (crossroads, tunnels), traffic lights and information boards. One of the goals may be to make the traffic fluent and prevent traffic jams in case of unexpected circumstances. The elements are observed (automatically or by humans) to provide information about the situation. One element may be observed by several supervisors corresponding to, e.g., different directions. The overall system is modeled as a single system including all dependences between the elements. The control decision is made based on all observations. Thus, although there are local supervisors that do not observe any problem, they need to react based on the observations of other supervisors, for instance, to close all the streets accessing a blocked tunnel.

There is an important motivation for decentralized supervisory control of discrete-event systems that do not a priori have a modular structure. It is well known that the abstraction of timed automata into region (zone) automata does not preserve the modular structure. Similarly, it is to be expected that the discretization of a hybrid system does not preserve its modular structure. It means that the original structure of a hybrid system is lost and we have to face the decentralized supervisory control problem instead of the modular supervisory control problem.

There are many different control policies based on two elementary ones: *conjunctive and permissive* (C & P) and *disjunctive and antipermissive* (D & A). For any decentralized control architecture, a corresponding notion of *coobservability* was proposed, which together with *controllability* form the necessary and sufficient conditions to achieve a specification by the controlled system. Nowadays, there are advanced architectures, such as an architecture with conditional decision (inferencing) [32, 54] or multi-level inferencing [23, 44]. A general approach consisting in several decentralized supervisory control architectures running in parallel was proposed by Chakib and Khoumsi [6].

Another approach to ensure coobservability is to extend locally observable events via communication among local supervisors. There exist decentralized control problems that cannot be solved without communication. Decentralized control with communicating supervisors, where an occurrence of transitions visible to one supervisor can be communicated to other supervisors, has been studied, e.g., in [1, 33, 37].

So far, almost all results available in the literature are only existential. There exist a few papers providing constructive results to compute a controllable and coobservable sublanguage of a specification, but the problem is in general computationally difficult. The existence of local supervisors enforcing the safety specification is decidable if nonblockingness is not required (e.g., for prefix-closed languages). However, if the marked language of the controlled system has to be included in the specification so that the controlled system is nonblocking, the existence of such local supervisors is undecidable [46, 47].

We focus on the computation of a controllable and coobservable sublanguage by using additional communications among the supervisors. Our study is restricted to the original (C & P) architecture [39, 53] and motivated by the relationship between decentralized and modular supervisory control and their key concepts: C & P coobservability and separability. This relationship is investigated in Komenda et al. [12], where the decentralized framework is plugged into the modular framework. The approach is based on the concurrent (separable) over-approximation of the plant. In decentralized control, there is no assumption on the plant. Therefore, both the system and the specification are replaced with their infimal separable superlanguages. However, in the likely case the specification fails to be separable, one only computes a solution for this new specification, which often fails to be included in the specification. Otherwise stated, it is assumed that the constructed sublanguage is included in the specification [12]. This is the point, where our approach comes into the picture. We overcome the problem of an inseparable specification by making it separable via communication (using the notion of *conditional decomposability* [15]). The approach of Komenda et al. [12] further requires *mutual controllability* to ensure coobservability of the separable over-approximation of the specification. This can be omitted in our approach, although we show that it is useful to ensure the centralized optimality.

Decomposable over-approximations were also considered by Jiang and Kumar [10], where decomposability of the specification is an additional assumption, whereas in our approach, it is enforced by the construction via additional communications.

In this paper, we construct a controllable and coobservable sublanguage of a specification with respect to possibly extended local observations (enriched by communication). The idea is different from the computation of purely coobservable sublanguages in the literature. First, we extend the set of locally observable events via communication of events observable by other supervisors. To compute such an extension, we use the technique for conditional decomposability [15]. Then we use the extended alphabets to compute the local supervisors in a fully decentralized way. We show that our construction guarantees coobservability (Theorem 7) as soon as the local supervisors are nonconflicting (in particular if they are prefix-closed). We then state two sufficient conditions under which the solution coincides with the centralized optimal solution. One approach uses mutual controllability (Theorem 9), which is a condition considering the structure of the plant, and the other the observer and LCC conditions (Theorem 10), which rather considers the structure of the supervisors. Since our approach is not restricted to prefix-closed languages, we further show how to handle the case of conflicting supervisors (Theorem 11). The complexity of our approach is briefly discussed in Section 6.3. We further show that separability with respect to the number of supervisors is PSPACE-complete (Theorem 5), which generalizes a result on the complexity of decomposability.

Although our approach is different from the existing approaches, the rough idea is analogous to that used for decentralized synthesis without communication [21, 43]. The approach of Kozák and Wonham [21] is similar to ours in that it computes fully decentralized solutions that guarantee coobservability. It essentially projects the centralized supervisor, whereas we compute local supervisors and coobservability is guaranteed by construction (from separability that is granted by distributed computation). The main difference is that we do not project the centralized supervisor, but rather the plant and the specification. Also, the condition of Takai [43] under which the fully decentralized supervisors achieve the centralized optimal solution (the centralized optimal supervisor must be observable with respect to all locally observable alphabets) is different from our condition that relies on structural properties of the plant: such as observer and local control consistency or mutual controllability and observability of projections of the plant.

Our approach is also related to the topic on sensor selection [11, 34, 48, 49], where there is a maximal observable event set and the sensors can be turned on and off. In our approach, we have static observations, rather than dynamic, and communicate them among supervisors, where needed. Our work can be extended to dynamic observations in the future.

This work is an extended version of the conference paper [13]. We extend our approach to non-prefixed-closed languages and to partial observations, compare it to the centralized optimal solution, and include a discussion on the complexity of related problems.

2 Preliminaries and Definitions

We assume that the reader is familiar with decentralized supervisory control of discrete-event systems [5, 52]. Let A be a finite nonempty set of events (an *alphabet*), and let A^* denote the set of all finite words over A ; the empty word is denoted by ε . A *language* over A is a subset of A^* . The *prefix closure* of a language L over A is the set $\bar{L} = \{w \in A^* \mid \text{there exists } u \in A^* \text{ such that } wu \in L\}$. A language L is *prefix-closed* if $L = \bar{L}$.

A *generator* is a quintuple $G = (Q, A, f, q_0, Q_m)$, where Q is a finite set of *states*, A is an alphabet (of *events*), $f : Q \times A \rightarrow Q$ is a *partial transition function*, $q_0 \in Q$ is the *initial state*, and $Q_m \subseteq Q$ is a set of *marked states*. The transition function f can be extended to the domain $Q \times A^*$ in the usual way. The language *generated* by G is the set

$L(G) = \{s \in A^* \mid f(q_0, s) \in Q\}$ and the language *marked* by G is the set $L_m(G) = \{s \in A^* \mid f(q_0, s) \in Q_m\}$. The paper is restricted to *regular languages*, that is, languages marked by a generator.

A *projection* $P_o : A^* \rightarrow A_o^*$ is a morphism defined by $P_o(a) = \varepsilon$, if $a \in A \setminus A_o$, and $P_o(a) = a$, if $a \in A_o$. It is extended (as a morphism for concatenation) from events to words by induction. The *inverse image* of P_o is defined as $P_o^{-1}(a) = \{s \in A^* \mid P_o(s) = a\}$. These definitions can naturally be extended to languages. In what follows, we use the notation P_v^u to denote a projection from A_u^* to A_v^* and we write simply P_v if $A_u = A$.

A *synchronous product* of languages $L_i \subseteq A_i^*$, for $i = 1, \dots, n$, is defined as $\|_{i=1}^n L_i = \bigcap_{i=1}^n P_i^{-1}(L_i) \subseteq A^* = (\bigcup_{i=1}^n A_i)^*$, where $P_i : A^* \rightarrow A_i^*$ is a projection. Languages L_i are *synchronously nonconflicting* if $\|_{i=1}^n L_i = \|_{i=1}^n \overline{L_i}$.

Let L be a prefix-closed language over A . A language $K \subseteq L$ is *observable* with respect to L and $P_o : A^* \rightarrow A_o^*$ if for all $s \in \overline{K}$ and $a \in A_c$, $sa \notin \overline{K}$ and $sa \in L$ implies that $P_o^{-1}P_o(s)\{a\} \cap \overline{K} = \emptyset$. Language K is *normal* with respect to L and P_o if $\overline{K} = P_o^{-1}P_o(\overline{K}) \cap L$.

3 Decentralized Supervisory Control

A *controlled generator* over an alphabet A is a structure $(G, (A_{c,i})_{i=1}^n, (A_{o,i})_{i=1}^n)$, where G is a generator over A , $A_{c,i} \subseteq A$ are sets of *locally controllable events* and $A_{o,i} \subseteq A$ are sets of *locally observable events*. We denote $A_c = \bigcup_{i=1}^n A_{c,i}$, $A_o = \bigcup_{i=1}^n A_{o,i}$, $A_u = A \setminus A_c$, and $A_{uo} = A \setminus A_o$. Projections to locally observable events are denoted by $P_{o,i} : A^* \rightarrow A_{o,i}^*$.

Let $\Gamma_i = \{\gamma \subseteq A \mid \gamma \supseteq (A \setminus A_{c,i})\}$ be a set of local control patterns. A supervisor S_i is a mapping $S_i : P_{o,i}(L(G)) \rightarrow \Gamma_i$, where $S_i(s)$ is the set of locally enabled events if S_i observes $s \in A_{o,i}^*$. The global control law S is the conjunction of local supervisors S_i given by $S(w) = \bigcap_{i=1}^n S_i(P_{o,i}(w))$ for $w \in A^*$. The *closed-loop system* is the smallest language $L(S/G)$ such that $\varepsilon \in L(S/G)$ and if $s \in L(S/G)$, $sa \in L(G)$ and $a \in S(s)$, then $sa \in L(S/G)$. Control objectives of decentralized control are defined using a specification language K . Let $L_m(S/G) = L(S/G) \cap K$. If $\overline{L_m(S/G)} = \overline{L(S/G)}$, the closed-loop system is called *nonblocking*. The goal of decentralized control is to find supervisors $(S_i)_{i=1}^n$ such that $L_m(S/G) = K$ and $\overline{L_m(S/G)} = \overline{K}$.

Necessary and sufficient conditions to achieve a specification by a joint action of local supervisors are controllability and coobservability [39]. Let L be a prefix-closed language over A . A language $K \subseteq L$ is *controllable* with respect to L and A_u if $\overline{K}A_u \cap L \subseteq \overline{K}$. A language $K \subseteq L$ is *coobservable* with respect to L and $(A_{o,i})_{i=1}^n$ if for all $s \in \overline{K}$, $a \in A_c$, and $sa \in L \setminus \overline{K}$, there exists $i \in \{1, 2, \dots, n\}$ such that $a \in A_{c,i}$ and $(P_{o,i}^{-1}(P_{o,i}(s))\{a\}) \cap \overline{K} = \emptyset$. Intuitively, if, after a word s from the specification, the extension by an event a is illegal (it does not exist in the specification but exists in the plant), then there must exist at least one local supervisor that can issue the decision “disable the event a ”.

The control law of local supervisors associated to the C & P architecture is called *permissive*, since the default action is to enable an event whenever a supervisor has an ambiguity what to do with it. Specifically, the control law of supervisor S_i on s is defined as $S_i(s) = (A \setminus A_{c,i}) \cup \{a \in A_{c,i} \mid \text{there exists } s' \in K \text{ with } P_{o,i}(s') = P_{o,i}(s) \text{ and } s'a \in K\}$. With the permissive local policy, we always achieve all words in the specification. The concern is then *safety*, expressed by coobservability.

Let $X \subseteq A$ be an alphabet. In the rest of the paper, we use the convention to define X_u as $X_u = X \cap A_u$. Similarly, $X_c = X \cap A_c$, $X_o = X \cap A_o$, and $X_{uo} = X \cap A_{uo}$.

4 Separability

A conceptually simpler condition than coobservability is known in the literature as *decomposability* [38]. A language K over A is *decomposable* with respect to $(A_i)_{i=1}^n$ and L if $K = (\bigcap_{i=1}^n P_i^{-1}P_i(K)) \cap L$, where $P_i : A^* \rightarrow A_i^*$. The inclusion $K \subseteq \bigcap_{i=1}^n P_i^{-1}P_i(K) \cap L$ holds true whenever $K \subseteq L$.

It was shown in Rudie and Wonham [39] that under mild and reasonable assumptions on the structure of locally controllable and locally observable events, decomposability is equivalent to coobservability. Equivalence holds if all locally controllable events are locally observable. Since we use only one implication, we can weaken the conditions.

A special case of decomposability for $L = A^*$ is known as *separability* [50]. In what follows, we have that $A = \bigcup_{i=1}^n A_i$, which allows us to replace intersection with parallel composition in the definition of decomposability (separability).

If local supervisors do not observe all events, that is, $A_{uo} \neq \emptyset$, we consider an arbitrary decomposition of A_{uo} into (not necessarily disjoint) sets $A_{uo,i}$, such that $A_{uo,i} \subseteq A_{uo}$ and $\bigcup_{i=1}^n A_{uo,i} = A_{uo}$. How to obtain such a decomposition is discussed at the end of the following subsection. The alphabet of supervisor S_i is then $A_i = A_{o,i} \cup A_{uo,i}$, where S_i still observes only the events from $A_{o,i}$.

Theorem 1. *Let L be a prefixed-closed language and assume that $A_{o,i} \cap A_c \subseteq A_{c,i}$, for $i = 1, \dots, n$. Let $K \subseteq L$ be such that $\overline{K} = \|_{i=1}^n \overline{K_i}$, where $K_i \subseteq A_i^*$. If*

1. *either K_i is normal with respect to $P_i(L)$ and $P_{o,i}^i$, for all $i = 1, \dots, n$,*
2. *or $A_c \subseteq A_o$ and K_i is observable with respect to $P_i(L)$ and $P_{o,i}^i$, for all $i = 1, \dots, n$,*

then K is coobservable with respect to $(A_{o,i})_{i=1}^n$ and L .

Proof. For the sake of contradiction, assume that \bar{K} is not coobservable with respect to $(A_{o,i})_{i=1}^n$ and L . Then there exist $s \in \bar{K}$ and $a \in A_c$ such that $sa \in L \setminus \bar{K}$ and for each $i \in \{1, 2, \dots, n\}$, either $a \notin A_{c,i}$ or $P_{o,i}^{-1}P_{o,i}(s)\{a\} \cap \bar{K} \neq \emptyset$. Let $t = P_i(s) \in P_i(\bar{K}) \subseteq \bar{K}_i$, then $sa \in L$ implies that $tP_i(a) \in P_i(L)$. We show below that $sa \in P_i^{-1}(\bar{K}_i)$. It then completes the proof, since $sa \in \cap_{i=1}^n P_i^{-1}(\bar{K}_i) = \bar{K}$, which is a contradiction with $sa \in L \setminus \bar{K}$.

If $a \in A_{c,i}$, then there exists s_i such that $s_i a \in \bar{K}$ and $P_{o,i}(s_i) = P_{o,i}(s)$. Let $t_i = P_i(s_i) \in P_i(\bar{K}) \subseteq \bar{K}_i$, then $s_i a \in \bar{K}$ implies that $t_i a \in P_i(\bar{K}) \subseteq \bar{K}_i$. Moreover, $P_{o,i}^i(t) = P_{o,i}^i(P_i(s)) = P_{o,i}(s) = P_{o,i}(s_i) = P_{o,i}^i(P_i(s_i)) = P_{o,i}^i(t_i)$. Since $t_i a \in \bar{K}_i$ and $t_i a \in (P_{o,i}^i)^{-1}P_{o,i}^i(t)\{a\}$, observability of K_i with respect to $P_i(L)$ and $P_{o,i}^i$ implies that $ta = P_i(sa) \in \bar{K}_i$, i.e., $sa \in P_i^{-1}(\bar{K}_i)$.

If $a \notin A_{c,i}$, then $a \notin A_{o,i} \cap A_c$. Since $a \in A_c$, $a \notin A_{o,i}$. If $a \notin A_{uo,i}$, then $a \notin A_i$, hence $P_i(a) = \varepsilon$ and $P_i(sa) = P_i(s) \in P_i(\bar{K}) \subseteq \bar{K}_i$ implies that $sa \in P_i^{-1}(\bar{K}_i)$. If $a \in A_{uo,i}$, then normality of K_i with respect to $P_i(L)$ implies that $\bar{K}_i = (P_{o,i}^i)^{-1}P_{o,i}^i(\bar{K}_i) \cap P_i(L)$. Since $t \in \bar{K}_i$ and $P_{o,i}^i(a) = \varepsilon$, we have that $ta \in (P_{o,i}^i)^{-1}P_{o,i}^i(\bar{K}_i)$. Then $ta \in P_i(L)$ implies that $ta \in \bar{K}_i$, and $ta = P_i(sa)$ then gives that $sa \in P_i^{-1}(\bar{K}_i)$. This completes the proof for K_i normal. If K_i is not normal, we have that $A_c \subseteq A_o$ and K_i is observable. Then $a \in A_c$ implies that $a \notin A_{uo}$, hence the only possible case is $a \notin A_{uo,i}$, which we have already shown above. \square

Note that the assumption $A_c \subseteq A_o$ in case 2 of Theorem 1 does not mean that $A_{c,i} \subseteq A_{o,i}$, but it only requires that every controllable event is observed by one of the supervisors.

5 Communication

If a language is not separable, it would be natural to take a maximal separable sublanguage. Unfortunately, this is not algorithmically possible [27]. To overcome this issue, we use the notion of *conditional decomposability* [15].

Definition 1 (Conditional decomposability). A language K is *conditionally decomposable* with respect to alphabets $(A_i)_{i=1}^n$ and Σ , where $\bigcup_{i \neq j} (A_i \cap A_j) \subseteq \Sigma$, if $K = \parallel_{i=1}^n P_{i+\Sigma}(K)$, where $P_{i+\Sigma} : A^* \rightarrow (A_i \cup \Sigma)^*$ is a projection.

Conditional decomposability requires to find another alphabet Σ containing all shared events such that K is separable with respect to $(A_i \cup \Sigma)_{i=1}^n$. The main advantage of conditional decomposability, compared to separability, is that every language can be made conditionally decomposable. This overcomes the undecidable issue of finding a nonempty separable sublanguage. Indeed, one could take $\Sigma = A$, but the aim is to find a reasonably small extension.

The main idea to compute the controllable and coobservable sublanguage discussed below is to over-approximate the plant language L by a new modular plant $\parallel_{i=1}^n P_{i+\Sigma}(L)$. It is better to consider the projections $P_{i+\Sigma}(L)$ rather than $P_i(L)$ used in Komenda et al. [12], because the larger the extension Σ , the better the over-approximation of L , as shown in the following lemma.

Lemma 2. Let $(A_i)_{i=1}^n$ be alphabets and L be a language over $A = \bigcup_{i=1}^n A_i$. Let $\Sigma \subseteq A$, and let $P_i : A^* \rightarrow A_i^*$ and $P_{i+\Sigma} : A^* \rightarrow (A_i \cup \Sigma)^*$ be projections. Then $L \subseteq \parallel_{i=1}^n P_{i+\Sigma}(L) \subseteq \parallel_{i=1}^n P_i(L)$.

Proof. The first inclusion holds for any projection.

To prove the other inclusion, notice that $P_{i+\Sigma}(L) \subseteq (P_i^{i+\Sigma})^{-1}(P_i^{i+\Sigma}(P_{i+\Sigma}(L))) = (P_i^{i+\Sigma})^{-1}(P_i(L))$, where $P_i^{i+\Sigma}$ is the projection P_i restricted to the domain $(A_i \cup \Sigma)^*$. Then $P_{i+\Sigma}^{-1}(P_{i+\Sigma}(L)) \subseteq P_{i+\Sigma}^{-1}(P_i^{i+\Sigma})^{-1}(P_i(L)) = P_i^{-1}P_i(L)$, which completes the proof. \square

It may seem that the largest Σ is the best choice. However, a larger Σ means more communication or more sensors to observe the system (the local supervisors need to observe more). On the other hand, to compute a minimal extension Σ with respect to the cardinality is NP-hard [17]. Nevertheless, there is an algorithm to find an acceptable extension in polynomial time [15]. In general, to find a suitable extension in a reasonable time is an interesting research topic. Notice that the choice of Σ can be influenced by several factors, such as the price of sensors, the (im)possibility to observe an event (by a specific local supervisor) etc.

Even if we consider the minimal extension Σ , it may happen that too many events are forced to be communicated between all supervisors though it is not needed. We demonstrate this in the following example. It shows that it is more convenient to search for some local alphabets so that the specification is separable with respect to them.

Example 1. Consider the language K over $A = \{a, b, c, d, e, f\}$, whose generator is depicted in Fig. 1, and the alphabets $A_{o,1} = \{a, e\}$, $A_{o,2} = \{b, e\}$, $A_{o,3} = \{c, f\}$, and $A_{o,4} = \{d, f\}$. To make K conditionally decomposable with respect to $(A_{o,i})_{i=1}^4$ and Σ , the extension Σ must contain at least all shared events, that is, e and f ; actually, $\Sigma = \{e, f, a, c\}$ is a minimal extension making K conditionally decomposable with respect to $(A_{o,i})_{i=1}^4$ and Σ . Notice, however, that a needs to be communicated only between the supervisors S_1 and S_2 , whereas c needs to be communicated only between the supervisors S_3 and S_4 . Specifically, K is conditionally decomposable with respect to alphabets $A_{o,1} \cup A_{o,2}$, $A_{o,3} \cup A_{o,4}$ and $\Sigma_{all} = \{f\}$. This means that $K = P_{\{a,b,e,f\}}(K) \parallel P_{\{c,d,f\}}(K)$. Having this, notice that $P_{\{a,b,e,f\}}(K) = \overline{\{abef, afb, fab\}}$ is conditionally decomposable with respect to $A_{o,1} \cup \Sigma_{all}$, $A_{o,2} \cup \Sigma_{all}$ and $\Sigma_{1,2} = \{a, e, f\}$, and $P_{\{c,d,f\}}(K) = \overline{\{cdf\}}$ is

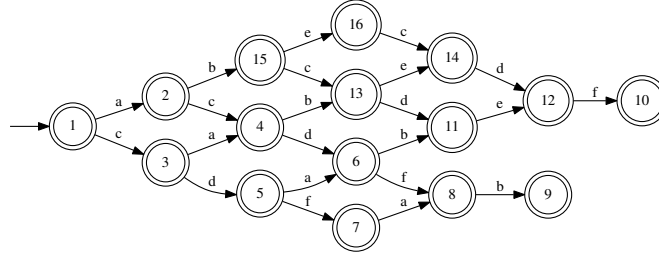


Figure 1: Specification K

conditionally decomposable with respect to $A_{o,3} \cup \Sigma_{all}$, $A_{o,4} \cup \Sigma_{all}$ and $\Sigma_{3,4} = \{c, f\}$. This means that a and c are communicated only locally and only f is communicated globally. In other words, K is separable with respect to alphabets $\{a, e, f\}$, $\{a, b, e, f\}$, $\{c, f\}$, $\{c, d, f\}$.

The reader can compare this with the purely conditional decomposable case computing the single extension $\Sigma = \{e, f, a, c\}$ making the language K separable with respect to $(A_{o,i} \cup \Sigma)_{i=1}^n$, that is, with respect to alphabets $\{a, c, e, f\}$, $\{a, b, c, e, f\}$, $\{a, c, e, f\}$, and $\{a, c, d, e, f\}$.

The following theorem is a generalization of a result obtained for pairwise disjoint alphabets in Willner and Heymann [50].

Theorem 3 ([15]). *Let K be a language, and let $(A_i)_{i=1}^n$ and Σ be alphabets such that $\bigcup_{i \neq j} (A_i \cap A_j) \subseteq \Sigma \subseteq \bigcup_{j=1}^n A_j$. To decide whether K is separable with respect to $(A_i \cup \Sigma)_{i=1}^n$ can be done in polynomial time.*

In Example 1 we need to check separability of K with respect to alphabets $A_{o,1} \cup A_{o,2} \cup \Sigma_{all} = \{a, b, e, f\}$ and $A_{o,3} \cup A_{o,4} \cup \Sigma_{all} = \{c, d, f\}$. According to Theorem 3, this can be done in polynomial time. Moreover, the alphabets Σ_{all} , $\Sigma_{1,2}$ and $\Sigma_{3,4}$ are computed in polynomial time. This suggests the following refinement of the conditional decomposability procedure.

Procedure: REFINED CONDITIONAL DECOMPOSABILITY (RCD)

Let $(A_i)_{i=1}^n$ be alphabets such that $A_i \neq A_j$ for $i \neq j$, and let K be a language over $\bigcup_{i=1}^n A_i$. We define the equivalence relation \sim as the minimal equivalence relation such that $A_i \sim A_j$ if $A_i \cap A_j \neq \emptyset$. The following steps of the procedure are demonstrated in Example 2 below.

1. Let $A_i / \sim = \{A_j \mid A_j \sim A_i, 1 \leq j \leq n\}$ be the equivalence class of alphabets equivalent to A_i . Let $B_i = \bigcup_{A_j \in A_i / \sim} A_j$ be the set of all events that appear in A_i / \sim . Let $\{B_{k_1}, \dots, B_{k_m}\} = \{B_1, \dots, B_n\}$, that is, we remove duplicates, hence we have $B_{k_i} \cap B_{k_j} = \emptyset$ for $k_i \neq k_j$.
2. We first compute a global extension Σ_{all} of events shared by the sets $(B_{k_i})_{i=1}^m$, which makes K conditionally decomposable with respect to $(B_{k_i})_{i=1}^m$ and Σ_{all} . This gives

$$K = \parallel_{i=1}^m P_{B_{k_i} \cup \Sigma_{all}}(K).$$

3. Then, for each local part A_{k_i} / \sim , $i = 1, 2, \dots, m$, we make the language $P_{B_{k_i} \cup \Sigma_{all}}(K)$ conditionally decomposable with respect to $C_{k_i} = \{A_j \cup \Sigma_{all} \mid A_j \in A_{k_i} / \sim\}$ and Σ_{k_i} . The set notation is used here to eliminate duplicates. If C_{k_i} is a singleton, we set $\Sigma_{k_i} = \Sigma_{all}$. This then gives that

$$P_{B_{k_i} \cup \Sigma_{all}}(K) = \parallel_{B \in C_{k_i}} P_{B \cup \Sigma_{k_i}}(K).$$

Notice that if $B, B' \in C_{k_i}$, then $\Sigma_{all} \subseteq B \cap B'$, which implies that $\Sigma_{all} \subseteq \Sigma_{k_i}$.

4. Finally, for every $A_j \in A_{k_i} / \sim$, we set $\Sigma_j = \Sigma_{k_i}$.

We now illustrate this procedure on a previous example.

Example 2. Consider Example 1. In this case, we have that $A_{o,1} \sim A_{o,2}$ and $A_{o,3} \sim A_{o,4}$, hence $A_1 / \sim = A_2 / \sim = \{A_{o,1}, A_{o,2}\}$ and $A_3 / \sim = A_4 / \sim = \{A_{o,3}, A_{o,4}\}$. It gives that $B_1 = B_2 = A_{o,1} \cup A_{o,2}$ and $B_3 = B_4 = A_{o,3} \cup A_{o,4}$. Then K is conditionally decomposable with respect to B_1 , B_3 and $\Sigma_{all} = \{f\}$. In other words, K is separable with respect to $B_1 \cup \Sigma_{all}$ and $B_3 \cup \Sigma_{all}$. The projections are $P_{B_1 \cup \Sigma_{all}} : A^* \rightarrow \{a, b, e, f\}^*$ and $P_{B_3 \cup \Sigma_{all}} : A^* \rightarrow \{c, d, f\}^*$. The language $P_{B_1 \cup \Sigma_{all}}(K) = \{abef, afb, fab\}$ is conditionally decomposable with respect to $A_{o,1} \cup \Sigma_{all}$, $A_{o,2} \cup \Sigma_{all}$ and $\Sigma_1 = \{a, e, f\}$, and the language $P_{B_3 \cup \Sigma_{all}}(K) = \{cdf\}$ is conditionally decomposable with respect to $A_{o,3} \cup \Sigma_{all}$, $A_{o,4} \cup \Sigma_{all}$ and $\Sigma_3 = \{c, f\}$. Finally, we set $\Sigma_2 = \Sigma_1$ and $\Sigma_4 = \Sigma_3$. Language K is then separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^4$, that is, with respect to alphabets $\{a, e, f\}$, $\{a, b, e, f\}$, $\{c, f\}$, $\{c, d, f\}$.

We now show that the procedure is correct.

Theorem 4. Let $(A_i)_{i=1}^n$ be alphabets, such that $A_i \neq A_j$, for $i \neq j$, and let K be a language over $\bigcup_{i=1}^n A_i$. Let $(\Sigma_i)_{i=1}^n$ be the extensions computed by procedure RCD. Then K is separable with respect to $(A_i \cup \Sigma_i)_{i=1}^n$.

Proof. Let $\{B_{k_1}, \dots, B_{k_m}\} = \{B_1, \dots, B_n\}$. Then $B_{k_i} \cap B_{k_j} = \emptyset$, for $k_i \neq k_j$, since B_{k_i} and B_{k_j} contain events of different classes. After the computation of Σ_{all} , $K = \big\|_{i=1}^m P_{B_{k_i} \cup \Sigma_{all}}(K)$. Then, for each C_{k_i} , $i = 1, \dots, m$, we compute Σ_{k_i} such that $P_{B_{k_i} \cup \Sigma_{all}}(K) = \big\|_{B \in C_{k_i}} P_{B \cup \Sigma_{k_i}}(K) = \big\|_{A_j \in A_{k_i} / \sim} P_{A_j \cup \Sigma_{all} \cup \Sigma_j}(K)$, since $\Sigma_j = \Sigma_{k_i}$ and the synchronous product operation is idempotent. Together, we obtain that $K = \big\|_{i=1}^m \big\|_{A_j \in A_{k_i} / \sim} P_{A_j \cup \Sigma_{all} \cup \Sigma_j}(K) = \big\|_{i=1}^n P_{A_i \cup \Sigma_{all} \cup \Sigma_i}(K)$. Finally, since $\Sigma_{all} \subseteq \Sigma_i$, for all i , K is separable with respect to $(A_i \cup \Sigma_i)_{i=1}^n$. \square

The procedure significantly depends on the computation and verification of conditional decomposability, which relies on the assumption that all shared events of the alphabets under (local) considerations are always included in Σ_{all} (resp. Σ_i). This then allows us to use Theorem 3 to check the property in polynomial time. If we drop the assumption on shared events, the problem becomes computationally more difficult:

Theorem 5. Let $(A_i)_{i=1}^n$ be alphabets and K be a (prefix-closed) language over their union. To decide whether K is separable with respect to $(A_i)_{i=1}^n$ is PSPACE-complete.

A proof of this result can be found in the appendix. Notice that separability of K with respect to $(A_i)_{i=1}^n$ is a special case of decomposability where L is universal. As a consequence we obtain that decomposability of K with respect to $(A_i)_{i=1}^n$ and L is PSPACE-complete even if $L = A^*$. This generalizes a result that can be derived from the literature. Namely, coobservability of K with respect to $(A_i)_{i=1}^n$ and L is known to be PSPACE-complete [36]. Under some assumptions [39, Proposition 4.3], decomposability is equivalent to coobservability. Since the reduction in Rohloff et al. [36] satisfies these assumptions, decomposability of K with respect to $(A_i)_{i=1}^n$ and L is PSPACE-complete. In those proofs, however, L is different from A^* , hence our theorem generalizes this result.

We now return to the discussion on page 3 above Theorem 1 on how to obtain a decomposition of A_{uo} into $A_{uo,i}$ if A_{uo} is nonempty. For every alphabet $A_{o,i}$, the procedure RCD computes an extension Σ_i , which can be decomposed into observable and unobservable events, i.e., $\Sigma_i = \Sigma_{o,i} \cup \Sigma_{uo,i}$, where $\Sigma_{uo,i} = \Sigma_i \cap A_{uo}$. We then set $A_i = (A_{o,i} \cup \Sigma_{o,i}) \cup \Sigma_{uo,i}$, where $A_{o,i} \cup \Sigma_{o,i}$ are the events observed by supervisor S_i extended with communications.

6 Computation of Coobservable Sublanguages

Consider the settings of decentralized control, and let $(\Sigma_i)_{i=1}^n$ be extensions of local alphabets $(A_{o,i})_{i=1}^n$, computed by the procedure RCD, such that the specification K is separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^n$.

We now apply results of modular control, where the status of an event is global. Namely, all shared events have the same status in all components where they appear. This is not in general the case in decentralized control. However, since every shared event appears in at least one Σ_i , the choice of $\Sigma_{uo,i}$ then ensures that the status of shared observable events is the same in all components where they appear. For controllable events, we assume in the rest of the paper that if a supervisor observes a controllable event, then it can also control it. Formally, we assume that $A_{o,i} \cap A_c \subseteq A_{c,i}$, which must also hold after the extension, that is, $(A_{o,i} \cup \Sigma_{o,i}) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i})$, for $i = 1, 2, \dots, n$. Therefore, we adapt the controllable status of events, if needed, to ensure this condition.

We now summarize all the assumptions and notation in the following definition.

Definition 2 (Local supervisors). Let $(G, (A_{c,i})_{i=1}^n, (A_{o,i})_{i=1}^n)$ be a controlled generator over an alphabet A . We denote $A_c = \bigcup_{i=1}^n A_{c,i}$, $A_o = \bigcup_{i=1}^n A_{o,i}$, $A_u = A \setminus A_c$, and $A_{uo} = A \setminus A_o$. Let $K \subseteq L = L(G)$ be a specification language over A , and let $(\Sigma_i)_{i=1}^n$ be extensions of local alphabets $(A_{o,i})_{i=1}^n$ computed by the procedure RCD such that K is separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^n$, where $A = \bigcup_{i=1}^n (A_{o,i} \cup \Sigma_i)$. Let $P_{i+\Sigma_i} : A^* \rightarrow (A_{o,i} \cup \Sigma_i)^*$ denote the corresponding projection. We assume that $(A_{o,i} \cup \Sigma_{o,i}) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i})$, where $\Sigma_{o,i} = \Sigma_i \cap A_o$ and $\Sigma_{uo,i} = \Sigma_i \setminus \Sigma_{o,i}$. Let \mathbf{R}_i be languages that are controllable with respect to $P_{i+\Sigma_i}(L)$ and $(A_{o,i} \cup \Sigma_i)_u = (A_{o,i} \cup \Sigma_i) \cap A_u$, and

1. either normal with respect to $P_{i+\Sigma_i}(L)$ and $A_{o,i} \cup \Sigma_{o,i}$
2. or observable with respect to $P_{i+\Sigma_i}(L)$ and $A_{o,i} \cup \Sigma_{o,i}$ and $A_c \subseteq A_o$

such that $\big\|_{i=1}^n \mathbf{R}_i \subseteq K$.

Notice that normality implies observability [5], hence every \mathbf{R}_i is observable. However, if there is an \mathbf{R}_i , which is observable and not normal, then we require that $A_c \subseteq A_o$.

There are many ways how to compute the languages \mathbf{R}_i discussed in the literature. Namely, in the case of full local observations, it is natural to define the language

$$\mathbf{R}_i = \sup C_{i+\Sigma_i} = \sup C(P_{i+\Sigma_i}(K), P_{i+\Sigma_i}(L), (A_{o,i} \cup \Sigma_i)_u) \quad (1)$$

denoting the supremal controllable sublanguage of $P_{i+\Sigma_i}(K)$ with respect to $P_{i+\Sigma_i}(L)$ and $(A_{o,i} \cup \Sigma_i)_u$. In the case of partial observations, we may define the language

$$\mathbf{R}_i = \sup \text{CN}(P_{i+\Sigma_i}(K), P_{i+\Sigma_i}(L), (A_{o,i} \cup \Sigma_i)_u, A_{o,i} \cup \Sigma_{o,i})$$

denoting the supremal controllable and normal sublanguage of $P_{i+\Sigma_i}(K)$ with respect to $P_{i+\Sigma_i}(L)$, $(A_{o,i} \cup \Sigma_i)_u$ and $A_{o,i} \cup \Sigma_{o,i}$, see [5, 2]. Similarly, if $A_c \subseteq A_o$, we can define \mathbf{R}_i as the supremal controllable and relatively observable sublanguage [4, 18], or we can use any of the methods to compute a controllable and observable sublanguage, see, e.g., [7, 45]. In these cases, we have that $\mathbf{R}_i \subseteq P_{i+\Sigma_i}(K)$, and separability of K than implies that $\|_{i=1}^n \mathbf{R}_i \subseteq K$ as required in Definition 2.

However, we do not restrict the language \mathbf{R}_i to be included in $P_{i+\Sigma_i}(K)$, which allows us to define \mathbf{R}_i in many different ways. For instance, we can define \mathbf{R}_i as the infimal controllable (and normal/observable) superlanguage of $P_{i+\Sigma_i}(K)$ with respect to $P_{i+\Sigma_i}(L)$ and $(A_{o,i} \cup \Sigma_i)_u$ (and $A_{o,i} \cup \Sigma_{o,i}$), see [5, 24, 25, 40]. We can indeed combine the approaches, that is, one of the \mathbf{R}_i can be computed as a sublanguage, another one as a superlanguage, etc. In this case, we do not get the assumption $\|_{i=1}^n \mathbf{R}_i \subseteq K$ by construction, but we need to check it, which is in general a PSPACE-complete problem.¹ On the other hand, the advantage it brings is a potentially better (larger) solution as depicted in Example 4 below.

Another problem with infimal controllable superlanguages is that they do not exist for general languages, but only for prefix-closed languages. This issue can be avoided by the following choice of \mathbf{R}_i based on the computation of prefix-closed superlanguages.

Lemma 6. *Consider Definition 2. Let T_i be the prefix-closed infimal controllable (and normal/observable) superlanguage of $P_{i+\Sigma_i}(K)$ with respect to $P_{i+\Sigma_i}(L)$ and $(A_{o,i} \cup \Sigma_i)_u$ (and $A_{o,i} \cup \Sigma_{o,i}$). Then $\mathbf{R}_i = P_{i+\Sigma_i}(K) \cup [T_i \setminus P_{i+\Sigma_i}(K)]$ is controllable (and normal/observable) with respect to $P_{i+\Sigma_i}(L)$ and $(A_{o,i} \cup \Sigma_i)_u$ (and $A_{o,i} \cup \Sigma_{o,i}$).*

Proof. We show that $\overline{\mathbf{R}_i} = T_i$. Since $P_{i+\Sigma_i}(K) \subseteq T_i$ and $\mathbf{R}_i \subseteq T_i$, we have that $\overline{\mathbf{R}_i} \subseteq T_i$. To show that $T_i \subseteq \overline{\mathbf{R}_i}$, let $w \in T_i$. If $w \notin P_{i+\Sigma_i}(K)$, then $w \in \mathbf{R}_i$ by definition. If $w \in P_{i+\Sigma_i}(K)$, then there exists v such that $wv \in P_{i+\Sigma_i}(K) \subseteq \mathbf{R}_i$, hence $w \in \overline{\mathbf{R}_i}$. \square

An important feature of our computation is that we automatically obtain a coobservable sublanguage.

Theorem 7. *Consider Definition 2. If the languages \mathbf{R}_i are synchronously nonconflicting (in particular, if they are prefix-closed), then $\mathbf{R} = \|_{i=1}^n \mathbf{R}_i$ is a sublanguage of K controllable with respect to L and A_u , and coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i})_{i=1}^n$.*

Proof. By definition, we have that $\mathbf{R} = \|_{i=1}^n \mathbf{R}_i \subseteq K$. By Lemmas 13 and 14 (Lemma 15) in the appendix, \mathbf{R} is controllable and normal (observable) with respect to $\|_{i=1}^n P_{i+\Sigma_i}(L)$. Since $L \subseteq \|_{i=1}^n P_{i+\Sigma_i}(L)$, it is also controllable and normal (observable) with respect to L . Because $(A_{o,i} \cup \Sigma_{o,i}) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i})$ by the assumption and \mathbf{R}_i are synchronously nonconflicting, that is, $\overline{\mathbf{R}} = \|_{i=1}^n \overline{\mathbf{R}_i}$, Theorem 1 implies that $\mathbf{R} = \|_{i=1}^n \mathbf{R}_i$ is coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i})_{i=1}^n$. \square

We now illustrate this approach on a simple example.

Example 3. Consider the languages $K = \{aa, ba, bbd, abc\}$ and $L = \{aac, abc, bac, bbd\}$ over $A = \{a, b, c, d\}$, and alphabets $A_{o,1} = A_{c,1} = \{a, c\}$ and $A_{o,2} = A_{c,2} = \{b, d\}$. Then K is not coobservable with respect to L and $(A_{o,i})_{i=1}^2$, because none of the supervisors is able to distinguish between ab and ba , where the continuation of ba by c within the plant leads outside the specification while the continuation of ab by c remains within the specification.

We compute the extensions $\Sigma_1 = \Sigma_2 \supseteq A_{o,1} \cap A_{o,2}$ by the procedure RCD such that K is separable with respect to $A_{o,1} \cup \Sigma_1$ and $A_{o,2} \cup \Sigma_2$. It is sufficient to take $\Sigma_1 = \Sigma_2 = \{b\}$, which needs to be communicated/observed by both supervisors. Since our system is with complete observations, we compute $\mathbf{R}_1 = \sup C_{1+\Sigma_1} = \{aa, abc, ba, bbd\}$ and $\mathbf{R}_2 = \sup C_{2+\Sigma_2} = \{bbd\}$ as supremal controllable sublanguages, cf. (1). By Theorem 7, we have that $\mathbf{R}_1 \parallel \mathbf{R}_2$ is coobservable with respect to L and the extended alphabets $\{a, b, c\}$ and $\{b, d\}$. Indeed, supervisor S_1 that exerts the control power over the event c is now able to distinguish between the words ab , after which c should be allowed, and ba , after which c should be disabled.

The following example now demonstrates the advantage of using infimal superlanguages in combination with supremal sublanguages.

Example 4. Let $L = \{ab, ba, bdau, dbau\}$ be a language over $A = \{a, b, d, u\}$, and consider the alphabets $A_{o,1} = \{a, u\}$, $A_{c,1} = \{a, d\}$, $A_{o,2} = \{b, u\}$, $A_{c,2} = \{b\}$. Let $K = \{ab, ba, bd, db\}$ be a specification. Then K is not coobservable with respect to L and $(A_{o,i})_{i=1}^2$ because, for $db \in K$, we have $a \in A_c$, $dba \in L \setminus K$, and $P_1(db)a = a \in K$ and $P_2(db)a = ba \in K$. Thus, none of the supervisors can disable a after the word db .

¹There is a simple reduction from the finite-state automata intersection problem: Given a set of deterministic finite automata $\{G_i\}_{i=1}^n$ over a common alphabet B , is $\bigcap_{i=1}^n L_m(G_i) = \emptyset$? The problem is PSPACE-complete [22]. Let $A \cap B = \emptyset$. It is not hard to see that $\bigcap_{i=1}^n L_m(G_i) = \emptyset$ if and only if $\bigcap_{i=1}^n (L_m(G_i) \cup K) \subseteq K$. The reduction is polynomial since every $L_m(G_i) \cup K$ is represented by a generator computed from G_i and the generator for K in polynomial time by the standard product construction.

First, we compute only sublanguages. Let $\Sigma_1 = \Sigma_2 = \{d, u\}$ be the extensions of local observations computed by RCD. Then K is separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^2$. Notice that $P_{1+\Sigma_1}(K) = \{a, d, \varepsilon\}$ and $P_{2+\Sigma_2}(K) = \{\overline{bd}, \overline{db}\}$, and that $P_{1+\Sigma_1}(L) = \{a, \overline{dau}\}$ and $P_{2+\Sigma_2}(L) = \{\overline{bdu}, \overline{dbu}\}$. Then $\mathbf{R}_1 = \sup C_{1+\Sigma_1} = P_{1+\Sigma_1}(K)$ and $\mathbf{R}_2 = \sup C_{2+\Sigma_2} = \{b, d, \varepsilon\}$, and the solution $\mathbf{R}_1 \parallel \mathbf{R}_2 \subsetneq K$ gives us a strict subset of K .

On the other hand, we can obtain the whole K if we consider infimal controllable superlanguages of $P_{i+\Sigma_i}(K)$ instead of supremal controllable sublanguages. Then we obtain the infimal controllable superlanguages $\mathbf{R}_1 = P_{1+\Sigma}(K) = \sup C_{1+\Sigma_1}$ and $\mathbf{R}_2 = P_{2+\Sigma}(K)$, and the solution $\mathbf{R}_1 \parallel \mathbf{R}_2 = K$ then gives us the whole specification as the resulting language.

6.1 Conditions for Optimality for Full Observations

We now present conditions under which the solution is optimal in the sense of maximal permissiveness, i.e., under which it coincides with the supremal centralized supervisor, if it exists. Since no centralized optimal solution exists in the case of partial observations, we focus only on the case of full observations, that is, $A_{uo} = \emptyset$. However, we point out that a similar result to Proposition 8 can be obtained using the notions of *mutual normality* [20, Theorem 4.23] or *mutual observability* [19, Theorem 5.1]. Theorem 9 below can then be modified in the corresponding way.

Prefix-closed languages L_i over A_i are *mutually controllable* [26] if $L_j(A_{i,u} \cap A_j) \cap P_j P_i^{-1}(L_i) \subseteq L_j$, for $i, j = 1, 2, \dots, n$. The following compatibility between supremal controllable sublanguages and the synchronous composition operator is known [26]. We state the result only for prefix-closed languages and refer the reader to Lee and Wong [26] for the conditions on general languages.

Proposition 8. *Assume that $A_{o,i} \cap A_c \subseteq A_{c,i}$. If the prefix-closed languages $L_i \subseteq A_i^*$ are mutually controllable, then $\|_{i=1}^n \sup C(P_i(K), P_i(L), A_{i,u}) = \sup C(\|_{i=1}^n P_i(K), \|_{i=1}^n P_i(L), A_u)$ holds true for any prefix-closed language $K \subseteq L$.*

It may be that $P_{i+\Sigma_i}(L)$ and $P_{j+\Sigma_j}(L)$ are mutually controllable for fairly small alphabets Σ_i and Σ_j . However, if we do not require K to be separable, we cannot guarantee that the resulting supremal controllable sublanguage is included in K . This is the main issue with the approach in Komenda et al. [12]. Therefore, in this paper, we compute the communications $(\Sigma_i)_{i=1}^n$ using the procedure RCD such that K is separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^n$ to ensure the inclusion of the resulting supremal controllable sublanguage in K . In addition, if L is also separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^n$, which can again be ensured in the same way as for K , we obtain the optimal centralized solution.

Theorem 9. *Consider Definition 2. Let $K \subseteq L$ be prefix-closed languages, and let K and L be separable with respect to $(A_{o,i} \cup \Sigma_i)_{i=1}^n$. Let $\sup C_{i+\Sigma_i}$ be defined as in (1). If $P_{i+\Sigma_i}(L)$ and $P_{j+\Sigma_j}(L)$ are mutually controllable, for $i, j = 1, 2, \dots, n$, then $\|_{i=1}^n \sup C_{i+\Sigma_i} = \sup C(K, L, A_u)$ is coobservable with respect to L and $(A_{o,i} \cup \Sigma_i)_{i=1}^n$.*

Proof. Since $\sup C_{i+\Sigma_i} \subseteq P_{i+\Sigma_i}(K)$, $\|_{i=1}^n \sup C_{i+\Sigma_i} \subseteq \|_{i=1}^n P_{i+\Sigma_i}(K) = K$ by separability of K . By Theorem 7, $\|_{i=1}^n \sup C_{i+\Sigma_i}$ is controllable with respect to $\|_{i=1}^n P_{i+\Sigma_i}(L) = L$ and A_u , hence coobservable with respect to L and $(A_{o,i} \cup \Sigma_i)_{i=1}^n$ by Theorem 1. Finally, by Proposition 8, $\|_{i=1}^n \sup C_{i+\Sigma_i} = \sup C(K, L, A_u)$. \square

Note that mutual controllability holds if all shared events are controllable.

Example 5. Consider Example 3. The only shared event between $A_{o,1} \cup \Sigma_1$ and $A_{o,2} \cup \Sigma_2$ is event b , which is controllable. Therefore, $P_{1+\Sigma_1}(L)$ and $P_{2+\Sigma_2}(L)$ are mutually controllable. By Theorem 9, $\sup C_{1+\Sigma_1} \parallel \sup C_{2+\Sigma_2} = \sup C(K, L, A_u)$ is coobservable with respect to L and the extended alphabets $\{a, b, c\}$ and $\{b, d\}$.

The assumption that all shared events are controllable is too strong. We now present a result that ensures optimality and is based on the notions of an L -observer and LCC.

A projection $P_k : A^* \rightarrow A_k^*$, for $A_k \subseteq A$, is an L -observer for a language $L \subseteq A^*$ if for all $s \in \bar{L}$, if $P_k(s)t \in P_k(L)$, then there exists $u \in A^*$ such that $su \in L$ and $P_k(u) = t$ [51, 3]. The co-domain of a projection can always be extended to fulfill the condition. Although to compute the minimal extension is NP-hard, there is a polynomial-time algorithm to find an acceptable extension [9, 30]. The property also prevents the state explosion when computing projections. If P is an L -observer, then the generator for $P(L)$ is not larger (usually much smaller) than the one for L .

Let L be a prefix-closed language over A , and let $A_k \subseteq A$. Projection $P_k : A^* \rightarrow A_k^*$ is *locally control consistent* (LCC) with respect to a word $s \in L$ if for all events $a_u \in A_k \cap A_u$ such that $P_k(s)a_u \in P_k(L)$, it holds that either there does not exist any word $u \in (A \setminus A_k)^*$ such that $sua_u \in L$, or there exists a word $u \in (A_u \setminus A_k)^*$ such that $sua_u \in L$. Projection P_k is LCC with respect to L if P_k is LCC for all words of L [42, 41]. Notice that LCC is a weaker condition than OCC defined in Zhong and Wonham [55].

Theorem 10. *Consider Definition 2. If every projection $P_{i+\Sigma_i}$ is an L -observer and LCC for L , and the languages $\sup C_{i+\Sigma_i}$ defined in (1) are synchronously nonconflicting (e.g., prefix-closed), then the language $\|_{i=1}^n \sup C_{i+\Sigma_i} = \sup C(K, L, A_u)$ is coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i})_{i=1}^n$.*

Proof. The identity $\|_{i=1}^n \sup C_{i+\Sigma_i} = \sup C(K, L, A_u)$ has been shown in the literature [8, 42]. Theorem 1 then finishes the proof. \square

Note that both properties L -observer and LCC can be ensured by further extending the alphabets Σ_i in polynomial time.

A similar result to Theorem 10 can be obtained for supremal controllable and normal sublanguages. A condition under which the parallel composition of local supremal controllable and normal languages equals to the global supremal controllable and normal sublanguage can be found in [14, Theorem 25].

6.2 Conflicting Supervisors

The remaining question is the case of conflicting local supervisors. It is PSPACE-complete to check whether a parallel composition (of an unspecified number) of generators is nonblocking [35]. However, this is a theoretical result and Malik [28] shows that current computers can explore more than 100 million of states using explicit algorithms without any optimization techniques. Moreover, it is possible to use an L -observer to alleviate the computational effort as used in the following construction.

Theorem 11. *Consider Definition 2. Let $L_C \subseteq \|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$ be a language that is controllable and normal (observable) with respect to $P_{\Sigma'}(L)$, Σ'_u and Σ'_o , where $\Sigma' \subseteq A$ contains all events shared by any pair of \mathbf{R}_i and \mathbf{R}_j , for $i \neq j$, and $P_{\Sigma'} : A^* \rightarrow \Sigma'^*$ is an \mathbf{R}_i -observer, for $i = 1, \dots, n$. If $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i} \cup \Sigma'_c)$, then $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is a sublanguage of K controllable (and normal/observable) with respect to L and A_u (and A_o) that is coobservable with respect to $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o)_{i=1}^n$, and whose components are synchronously nonconflicting.*

Proof. By definition and Lemma 17, we have that $L_C \subseteq \|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i) = P_{\Sigma'}(\|\|_{i=1}^n \mathbf{R}_i) \subseteq P_{\Sigma'}(K)$. Thus, $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C) = (\|\|_{i=1}^n \mathbf{R}_i) \parallel L_C \subseteq K \parallel P_{\Sigma'}(K) = K$.

To prove nonconflictiness, we make use of Lemma 16 in the appendix, which states that $\overline{\mathbf{R}_i \parallel L_C} = \overline{\mathbf{R}_i} \parallel \overline{L_C}$ if and only if $\overline{P_{\Sigma'}(\mathbf{R}_i) \parallel L_C} = \overline{P_{\Sigma'}(\mathbf{R}_i)} \parallel \overline{L_C}$. The second equation holds, because both sides are equal to $\overline{L_C}$. Using Lemma 16 again, we have that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C) = \|\|_{i=1}^n \mathbf{R}_i \parallel L_C$ if and only if $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i \parallel L_C) = \|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i) \parallel L_C$. Lemma 16 can be applied again, since $P_{\Sigma'}$ is an $(\mathbf{R}_i \parallel L_C)$ -observer. It follows from Theorem 2 in Pena et al. [29] saying that a composition of observers is an observer – note that $P_{\Sigma'}$ is an \mathbf{R}_i -observer by assumption and an L_C -observer since it is an identity. Again, the latter equation holds, because $P_{\Sigma'}(\mathbf{R}_i \parallel L_C) = P_{\Sigma'}(\mathbf{R}_i) \parallel L_C = L_C$, by Lemma 17 and the definition of L_C . Thus, both sides are equal to $\overline{L_C}$. To summarize, $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C) = \|\|_{i=1}^n \mathbf{R}_i \parallel L_C = \|\|_{i=1}^n \overline{\mathbf{R}_i} \parallel \overline{L_C}$.

To prove controllability (and normality/observability), note that \mathbf{R}_i is controllable (and normal/observable) with respect to $P_{i+\Sigma_i}(L)$, and L_C is controllable (and normal/observable) with respect to $P_{\Sigma'}(L)$. By Lemma 13 (Lemmas 14 and 15) in the appendix and the nonconflictiness shown above, $\mathbf{R}_i \parallel L_C$ is controllable (and normal/observable) with respect to $P_{i+\Sigma_i}(L) \parallel P_{\Sigma'}(L)$ and $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o) \cap A_u$. Similarly, $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is controllable (and normal/observable) with respect to $\|\|_{i=1}^n P_{i+\Sigma_i}(L) \parallel P_{\Sigma'}(L)$ and A_u . Since $L \subseteq \|\|_{i=1}^n P_{i+\Sigma_i}(L)$ and $L \subseteq P_{\Sigma'}^{-1}P_{\Sigma'}(L)$, we have that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is controllable (and normal/observable) with respect to L and A_u .

Since $\mathbf{R}_i \parallel L_C$ is normal (observable) with respect to $P_{i+\Sigma_i}(L) \parallel P_{\Sigma'}(L)$, and it holds that $P_{i+\Sigma_i+\Sigma'}(L) \subseteq P_{i+\Sigma_i}(L) \parallel P_{\Sigma'}(L)$, we have that $\mathbf{R}_i \parallel L_C$ is normal (observable) with respect to $P_{i+\Sigma_i+\Sigma'}(L)$. Thus, using Theorem 1, $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o)_{i=1}^n$. In general, to apply Theorem 1, we need to adjust the controllable status of events, if needed, to satisfy $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i} \cup \Sigma'_c)$. \square

In case of full observations, we strengthen the previous result by computing the language L_C as a sublanguage of $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$ controllable with respect to $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$ rather than to $P_{\Sigma'}(L)$, which may result in a larger language L_C because $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i) \subseteq P_{\Sigma'}(L)$.

Theorem 12. *Consider Definition 2. Let L_C be the supremal sublanguage of $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$ that is controllable with respect to $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$ and Σ'_u , where $\Sigma' \subseteq A$ contains all events shared by any pair of \mathbf{R}_i and \mathbf{R}_j , for $i \neq j$, and $P_{\Sigma'} : A^* \rightarrow \Sigma'^*$ is an \mathbf{R}_i -observer, for $i = 1, \dots, n$. If every projection $P_{i+\Sigma_i}$ is an L -observer and LCC for L , and the projection $P_{\Sigma'}$ is LCC for $\|\|_{i=1}^n \mathbf{R}_i$, then $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C) = \sup C(K, L, A_u)$ is coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o)_{i=1}^n$, whose components are synchronously nonconflicting. It is again under the assumption that $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i} \cup \Sigma'_c)$.*

Proof. Let $\sup C = \sup C(K, L, A_u)$. We first show that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is a subset of $\sup C$ coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o)_{i=1}^n$, and its components are synchronously nonconflicting. Similarly as in the proof of Theorem 11, we can show that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C) \subseteq K$ and that $\overline{\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)} = \|\|_{i=1}^n \overline{(\mathbf{R}_i \parallel L_C)} = \|\|_{i=1}^n \overline{\mathbf{R}_i} \parallel \overline{L_C}$.

To prove controllability of $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ with respect to L and A_u , note that \mathbf{R}_i is controllable with respect to $\overline{\mathbf{R}_i}$ and $(A_{o,i} \cup \Sigma_{o,i})_u$, and L_C is controllable with respect to $\|\|_{j=1}^n P_{\Sigma'}(\mathbf{R}_j)$ and Σ'_u . By Lemma 13 in the appendix and the nonconflictiness of \mathbf{R}_i and L_C shown above, $\mathbf{R}_i \parallel L_C$ is controllable with respect to $\overline{\mathbf{R}_i} \parallel (\|\|_{j=1}^n P_{\Sigma'}(\mathbf{R}_j))$ and $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o)_u$. Using the same argument on $\mathbf{R}_i \parallel L_C$, for $i = 1, \dots, n$, we obtain that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is controllable with respect to $\|\|_{i=1}^n \overline{(\mathbf{R}_i \parallel L_C)} = \|\|_{i=1}^n \overline{\mathbf{R}_i} \parallel \overline{L_C}$ and A_u . One more application of Lemma 13 on $\overline{\mathbf{R}_i}$ gives that $\|\|_{i=1}^n \overline{\mathbf{R}_i}$ is controllable with respect to $\|\|_{i=1}^n P_{i+\Sigma_i}(L)$, hence with respect to $L \subseteq \|\|_{i=1}^n P_{i+\Sigma_i}(L)$, and A_u . Using transitivity of controllability, Lemma 18 in the appendix, we obtain that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is controllable with respect to L and A_u .

Since $\mathbf{R}_i \parallel L_C$ is trivially observable, Theorem 1 implies that $\|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$ is coobservable with respect to L and $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o)_{i=1}^n$. It again holds under the assumption that $(A_{o,i} \cup \Sigma_{o,i} \cup \Sigma'_o) \cap A_c \subseteq (A_{c,i} \cup \Sigma_{c,i} \cup \Sigma'_c)$.

It remains to show the other inclusion $\sup C \subseteq \|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$. We show it in two steps. First we show that $\sup C \subseteq \|\|_{i=1}^n \mathbf{R}_i$ and then that $P_{\Sigma'}(\sup C) \subseteq L_C$.

To show that $\sup C \subseteq \|\|_{i=1}^n \mathbf{R}_i$, we prove that, for every i , $P_{i+\Sigma_i}(\sup C) \subseteq \mathbf{R}_i$ by showing that $P_{i+\Sigma_i}(\sup C) \subseteq P_{i+\Sigma_i}(K)$ is controllable with respect to $P_{i+\Sigma_i}(L)$. To this end, let $t \in P_{i+\Sigma_i}(\sup C)$, $a \in (A_{o,i} \cup \Sigma_i) \cap A_u$, and $ta \in P_{i+\Sigma_i}(L)$. Then, there exists $s \in \sup C$ such that $P_{i+\Sigma_i}(s) = t$. Since $P_{i+\Sigma_i}$ is an L -observer, there exists $v \in A^*$ such that $sv \in L$ and $P_{i+\Sigma_i}(sv) = ta$, that is, $v = ua$ for some $u \in (A \setminus (A_{o,i} \cup \Sigma_i))^*$. The LCC property of $P_{i+\Sigma_i}$ for L and $sua \in L$ imply that there exists $u' \in (A_u \setminus (A_{o,i} \cup \Sigma_i))^*$ such that $su'a \in L$. Since u' is uncontrollable, controllability of $\sup C$ with respect to L and A_u implies that $su'a \in \sup C$, that is, $P_{i+\Sigma_i}(su'a) = ta \in P_{i+\Sigma_i}(\sup C)$. Thus, $P_{i+\Sigma_i}(\sup C)$ is controllable with respect to $P_{i+\Sigma_i}(L)$, hence $P_{i+\Sigma_i}(\sup C) \subseteq \mathbf{R}_i$.

Finally, we show that $P_{\Sigma'}(\sup C) \subseteq L_C$, i.e., that it is a subset of $\|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$ controllable with respect to $\|\|_{i=1}^n \overline{P_{\Sigma'}(\mathbf{R}_i)}$ and Σ'_u . Since $P_{i+\Sigma_i}(\sup C) \subseteq \mathbf{R}_i$, we have that $\sup C \subseteq \|\|_{i=1}^n P_{i+\Sigma_i}(\sup C) \subseteq \|\|_{i=1}^n \mathbf{R}_i$. Applying projection $P_{\Sigma'}$, we obtain $P_{\Sigma'}(\sup C) \subseteq P_{\Sigma'}(\|\|_{i=1}^n \mathbf{R}_i) = \|\|_{i=1}^n P_{\Sigma'}(\mathbf{R}_i)$, where the last equality is by Lemma 17, see the appendix. Using Theorem 2 in Pena et al. [29] saying that a composition of observers is an observer, the assumption on $P_{\Sigma'}$ to be an \mathbf{R}_i -observer, which also means that $P_{\Sigma'}$ is an $\overline{\mathbf{R}_i}$ -observer, implies that $P_{\Sigma'}$ is a $(\|\|_{i=1}^n \overline{\mathbf{R}_i})$ -observer. We show that $P_{\Sigma'}(\sup C)$ is controllable with respect to $\|\|_{i=1}^n \overline{P_{\Sigma'}(\mathbf{R}_i)}$ and Σ'_u . To this end, let $t \in P_{\Sigma'}(\sup C)$, $a \in \Sigma'_u$, and $ta \in \|\|_{i=1}^n \overline{P_{\Sigma'}(\mathbf{R}_i)} = P_{\Sigma'}(\|\|_{i=1}^n \overline{\mathbf{R}_i})$. Then, there exists $s \in \sup C$ such that $P_{\Sigma'}(s) = t$. Since $P_{\Sigma'}$ is a $(\|\|_{i=1}^n \overline{\mathbf{R}_i})$ -observer, there exists $v \in A^*$ such that $sv \in \|\|_{i=1}^n \overline{\mathbf{R}_i}$ and $P_{\Sigma'}(sv) = ta$, that is, $v = ua$ for some $u \in (A \setminus \Sigma')^*$. The LCC property of $P_{\Sigma'}$ for $\|\|_{i=1}^n \overline{\mathbf{R}_i}$ and $sua \in \|\|_{i=1}^n \overline{\mathbf{R}_i}$ then imply that there exists $u' \in (A_u \setminus \Sigma')^*$ such that $su'a \in \|\|_{i=1}^n \overline{\mathbf{R}_i}$. Since $\|\|_{i=1}^n \overline{\mathbf{R}_i} \subseteq L$, and u' is uncontrollable, controllability of $\sup C$ with respect to L and A_u implies that $su'a \in \sup C$. That is, $P_{\Sigma'}(su'a) = ta \in P_{\Sigma'}(\sup C)$, hence $P_{\Sigma'}(\sup C) \subseteq L_C$.

Together, $\sup C = \|\|_{i=1}^n \mathbf{R}_i \parallel L_C = \|\|_{i=1}^n (\mathbf{R}_i \parallel L_C)$, which completes the proof. \square

Notice that it is sufficient to require that $P_{\Sigma'}$ is an L -observer and LCC for L .

In case of partial observations, one could obtain the global supremal controllable and normal sublanguage in a similar way under the assumptions similar to those discussed below Theorem 10. Since, as already mentioned, there is no global optimal solution in the case of partial observations, we do not discuss this case in more detail.

6.3 Complexity

We now briefly discuss the complexity of our approach. Since we use standard notions, the complexity mainly depends on the complexity of corresponding algorithms for the computation of controllable and observable languages.

In Definition 2, we assume that K is separable. If this is not the case, a polynomial-time algorithm [15] is used to find extensions $(\Sigma_i)_{i=1}^n$ making the language separable with respect to extended alphabets. Then we compute local supervisors \mathbf{R}_i using the standard algorithms discussed below Definition 2. This requires to compute the projections of K and L , which is exponential in the worst case. However, the alphabet Σ_i can be chosen so that the projection $P_{i+\Sigma_i}$ is K - and L -observer, which then results in the computation of those projections in polynomial time [51]. The computation of \mathbf{R}_i is then performed in the respective time. It is polynomial in the case of full observations. To check/ensure optimality in the case of full observations, we either check whether the polynomially many pairs of languages are mutually controllable, which can be done in polynomial time, or further extend the alphabets Σ_i in polynomial time so that the observer and LCC conditions of Theorem 10 are satisfied. Furthermore, Theorems 7 and 10 require that the computed supervisors are nonconflicting. This is a PSPACE-complete problem [35]. However, this test can be skipped and we can directly compute the language L_C from Theorem 11, which may, in the worst case, require exponential space with respect to the number of local supervisors.

7 Conclusion

In this paper, we have shown how to construct a solution to the decentralized control problem (a controllable and coobservable sublanguage of a specification) by using additional communications. Our approach relies on the notion of conditional decomposability recently studied by the authors, which overcomes the undecidable problem to find a separable sublanguage of the specification. The computation of local supervisors is fully decentralized and coobservability is guaranteed by construction. We discuss two ways how to obtain the globally optimal solution in case of full observations if the computed languages are synchronously nonconflicting (prefix-closed). One is based on the notion of mutual controllability, the other on increasing the communication between supervisors. Indeed, both approaches can be combined as preferred. Our approach can be used for both prefix-closed and non-prefix-closed specifications. For conflicting supervisors, we showed how to impose nonconflictiness, hence coobservability.

A Proof of Theorem 5

Theorem 5. *Let $(E_i)_{i=1}^n$ be alphabets and K be a (prefix-closed) language over their union. To decide whether K is separable with respect to $(E_i)_{i=1}^n$ is PSPACE-complete.*

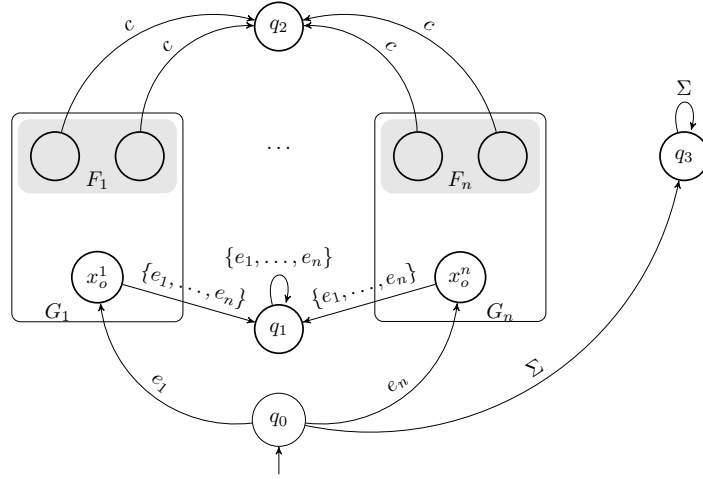


Figure 2: The automaton H

Proof. Standard techniques simulating a product automaton on-the-fly show that it belongs to PSPACE. To prove hardness, we reduce the finite-state automata intersection problem (INT): Given a set of deterministic finite automata $\{G_i\}_{i=1}^n$ over a common alphabet Σ . Is $\bigcap_{i=1}^n L_m(G_i) = \emptyset$? The problem is PSPACE-complete [22].

Let a set of deterministic automata $\{G_i\}_{i=1}^n$ with a common alphabet Σ be an instance of INT. Without loss of generality, we assume that $n \geq 3$, since if n is constant, then the problem is solvable in PTIME. Let $G_i = (X_i, \Sigma, \delta_i, x_o^i, F_i)$ and assume that all states of G_i are reachable from the initial state x_o^i . This assumption does not change the complexity. We construct a deterministic automaton H and alphabets $(E_i)_{i=1}^n$ in polynomial time such that $L(H)$ is separable if and only if $\bigcap_{i=1}^n L_m(G_i) = \emptyset$.

To this end, we define the automaton $H = (X, E, \delta, q_0, X)$ so that the set of states is $X = \bigcup_{i=1}^n X_i \cup \{q_0, q_1, q_2, q_3\}$, where q_0, q_1, q_2, q_3 are new states, $E = \Gamma \cup \Sigma$, where $\Gamma = \{e_1, \dots, e_n, c\}$ is an alphabet such that $\Gamma \cap \Sigma = \emptyset$, and the transition function is defined as follows. The initial state q_0 goes under e_i to the initial state x_o^i of G_i , $i = 1, \dots, n$, and for every a in Σ , q_0 goes under a to q_3 . In q_3 , there is a self-loop under every a in Σ . The transitions inside every G_i are unchanged. For every e in $\{e_1, \dots, e_n\}$ and every $i = 1, \dots, n$, we add a transition from x_o^i to q_1 under e . State q_1 contains a self-loop for every e in $\{e_1, \dots, e_n\}$. Finally, for $i = 1, \dots, n$, we add a c -transition from all states of F_i of G_i to state q_2 , cf. Fig. 2.

To complete the reduction, we define $E_i = E \setminus \{e_i\}$, which defines the projection $P_i : E^* \rightarrow E_i^*$, $i = 1, \dots, n$. Note that the reduction is polynomial. We show that $L(H)$ is separable with respect to $(E_i)_{i=1}^n$ if and only if $\bigcap_{i=1}^n L_m(G_i) = \emptyset$.

Assume that $t \in \bigcap_{i=1}^n L_m(G_i)$. Then $e_i t c \in L(H)$, for all $i = 1, \dots, n$, which implies that $t c \in P_i^{-1}(P_i(e_i t c))$, hence $t c \in \bigcap_{i=1}^n P_i(L(H))$. However, $t c \notin L(H)$, which shows that $L(H)$ is not separable with respect to $(E_i)_{i=1}^n$.

To prove the other direction, we assume that $L(H)$ is not separable and show that $\bigcap_{i=1}^n L_m(G_i) \neq \emptyset$. Let $w \in \bigcap_{i=1}^n P_i(L(H))$ and $w \notin L(H)$. Note that $\Sigma^* \cup \{e_1, \dots, e_n\}^* \subseteq L(H)$ because $L(H) = \Sigma^* \cup \{e_1, \dots, e_n\}^* \cup \bigcup_{i=1}^n e_i L_m(G_i) c \cup \bigcup_{i=1}^n e_i L(G_i)$. Therefore, for $i = 1, \dots, n$, w belongs to $P_i^{-1}(P_i(L(H))) = (\Sigma \cup \{e_i\})^* \cup \{e_1, \dots, e_n\}^* \cup P_i^{-1}(L_m(G_i) c) \cup \bigcup_{j \neq i} P_i^{-1}(e_j L_m(G_j) c) \cup P_i^{-1}(L(G_i)) \cup \bigcup_{j \neq i} P_i^{-1}(e_j L(G_j))$.

We first show that if c does not appear in w , then w belongs to $L(H)$. In this case, based on the above observation, w must contain at least one event from Σ and at least one event from $\{e_1, \dots, e_n\}$. Thus,

$$w \in (\Sigma \cup \{e_i\})^* \cup \bigcup_{j \neq i} P_i^{-1}(e_j L(G_j))$$

for $i = 1, \dots, n$, because $P_i^{-1}(L(G_i)) \subseteq (\Sigma \cup \{e_i\})^*$. Then $w \in \bigcap_{i=1}^n T_i$, where T_i is one of the languages forming the union above.

If, for some $i \neq j$, $T_i = (\Sigma \cup \{e_i\})^*$ and $T_j = (\Sigma \cup \{e_j\})^*$, then $w \in \Sigma^* \subseteq L(H)$; a contradiction.

If there is only one i such that $T_i = (\Sigma \cup \{e_i\})^*$ and, for all $j \neq i$, $T_j = P_j^{-1}(e_{k_j} L(G_{k_j}))$, where $k_j \neq j$, then w belongs to $\bigcap_{\ell=1}^n T_\ell$ if and only if $e_{k_j} = e_i$, for all $j \neq i$. Hence, for $j \neq i$, $T_i \cap T_j = e_i L(G_i)$, which implies that $\bigcap_{\ell=1}^n T_\ell = e_i L(G_i) \subseteq L(H)$; a contradiction.

The last option is that, for all i , $T_i = P_i^{-1}(e_{j_i} L(G_{j_i}))$, where $j_i \neq i$. Then there exist $j_k \neq j_\ell$ such that $e_{j_k} \neq e_{j_\ell}$. Without loss of generality, we assume that $w = v_1 e_{j_k} v_2 e_{j_\ell} w'$, where $v_1 v_2 w' \in (\Sigma \cup \{e_1, \dots, e_n\})^*$. Since $w \in T_k = P_k^{-1}(e_{j_k} L(G_{j_k}))$, $P_k(w) \in e_{j_k} L(G_{j_k})$, hence $P_k(v_1) = \varepsilon$ and $P_k(v_2 e_{j_\ell} w') \in \Sigma^*$, which implies that $j_\ell = k$, $v_1 \in \{e_k\}^*$ and $v_2 w' \in (\Sigma \cup \{e_k\})^*$. Similarly, $P_\ell(w) \in e_{j_\ell} L(G_{j_\ell})$ implies that $j_k = \ell$, $v_1 v_2 \in \{e_\ell\}^*$ and $w' \in (\Sigma \cup \{e_\ell\})^*$. Together, $v_1 v_2 = \varepsilon$, $w' \in \Sigma^*$, and $w = e_\ell e_k w'$, for $k \neq \ell$. By the assumption, there is a projection P_m such that $P_m \notin \{P_k, P_\ell\}$. Since $w \in T_m$, $P_m(w) \in (\Sigma \cup \{e_{j_m}\})^*$, and $P_m(e_\ell) = \varepsilon$ or $P_m(e_k) = \varepsilon$. The first case gives that $P_m = P_\ell$, the second that $P_m = P_k$, which is a contradiction.

Thus, we have show that if c does not appear in w , then w belongs to $L(H)$.

Assume that c appears in w . By the analysis above, $w \in \bigcap_{i=1}^n [P_i^{-1}(L_m(G_i)c) \cup \bigcup_{j \neq i} P_i^{-1}(e_j L_m(G_j)c)]$. It implies that there is exactly one c in w . Again, $w \in \bigcap_{i=1}^n T_i$, where T_i is one of the elements of the union.

Analogously as above, if, for all i , $T_i = P_i^{-1}(e_{j_i} L_m(G_{j_i})c)$, where $j_i \neq i$, then there are $j_k \neq j_\ell$ such that $e_{j_k} \neq e_{j_\ell}$. Without loss of generality, let $w = v_1 e_{j_k} v_2 e_{j_\ell} w' c w''$, where $v_1 v_2 w' w'' \in (\Sigma \cup \{e_1, \dots, e_n\})^*$. Then $P_k(w) \in e_{j_k} L_m(G_{j_k})c$, hence we have that $P_k(v_1) = \varepsilon$ and $P_k(v_2 e_{j_\ell} w' w'') \in \Sigma^*$, which implies that $j_\ell = k$, $v_1 \in \{e_k\}^*$ and $v_2 w' w'' \in (\Sigma \cup \{e_k\})^*$. Similarly, $P_\ell(w) \in e_{j_\ell} L_m(G_{j_\ell})c$ implies that $j_k = \ell$, $v_1 v_2 \in \{e_\ell\}^*$ and $w' w'' \in (\Sigma \cup \{e_\ell\})^*$. Together, $v_1 v_2 = \varepsilon$, $w' w'' \in \Sigma^*$, and $w = e_\ell e_k w' c w''$, for $k \neq \ell$. Let $P_m \notin \{P_k, P_\ell\}$ be a projection. Since $P_m(w) \in (\Sigma \cup \{e_{j_m}, c\})^*$, $P_m(e_\ell) = \varepsilon$ or $P_m(e_k) = \varepsilon$. The first case gives that $P_m = P_\ell$, the second that $P_m = P_k$, which is a contradiction.

Thus, there must exist i such that $T_i = P_i^{-1}(L_m(G_i)c)$. Then $P_i(w) \in L_m(G_i)c$, which implies that $w \in (\Sigma \cup \{e_i, c\})^*$. This means that, for $j \neq i$, $P_j(w) = w \in L_m(G_j)c \cup e_i L_m(G_i)c$. If $T_j = e_i L_m(G_i)c$, for some $j \neq i$, then $w \in (\Sigma \cup \{e_i, c\})^* \cap e_i L_m(G_i)c = e_i L_m(G_i)c \subseteq L(H)$; a contradiction again. Thus, it must be that for every $j \neq i$, $T_j = L_m(G_j)c$. Then $w \in \bigcap_{i=1}^n T_i = \bigcap_{i=1}^n L_m(G_i)c$ implies that $\bigcap_{i=1}^n L_m(G_i) \neq \emptyset$. \square

B Auxiliary results

Lemma 13 ([8]). *For $i = 1, 2$, let L_i be a prefix-closed language over A_i , and let $K_i \subseteq L_i$ be controllable with respect to L_i and $A_{i,u}$. Let $A = A_1 \cup A_2$. If K_1 and K_2 are synchronously nonconflicting, then $K_1 \parallel K_2$ is controllable with respect to $L_1 \parallel L_2$ and A_u .*

Lemma 14. *For $i = 1, 2$, let L_i be a prefix-closed language over A_i , and let $K_i \subseteq L_i$ be normal with respect to L_i , $A_{i,u}$ and $P_{o,i} : A_i^* \rightarrow A_{o,i}^*$. Let $A = A_1 \cup A_2$. If K_1 and K_2 are synchronously nonconflicting, then $K_1 \parallel K_2$ is normal with respect to $L_1 \parallel L_2$, A_u and $P_o : A^* \rightarrow A_o^*$.*

Proof. $P_o^{-1} P_o(\overline{K_1 \parallel K_2}) \cap L_1 \parallel L_2 \subseteq P_{o,1}^{-1} P_{o,1}(\overline{K_1}) \parallel P_{o,2}^{-1} P_{o,2}(\overline{K_2}) \parallel L_1 \parallel L_2 = \overline{K_1} \parallel \overline{K_2} = \overline{K_1 \parallel K_2}$. As the other inclusion always holds, the proof is complete. \square

Lemma 15. *For $i = 1, 2$, let L_i be a prefix-closed language over A_i , and let $K_i \subseteq L_i$ be observable with respect to L_i , $A_{i,u}$ and $P_{o,i} : A_i^* \rightarrow A_{o,i}^*$. Let $A = A_1 \cup A_2$. If K_1 and K_2 are synchronously nonconflicting, then $K_1 \parallel K_2$ is observable with respect to $L_1 \parallel L_2$, A_u and $P_o : A^* \rightarrow A_o^*$.*

Proof. Let $s, s' \in A^*$ be such that $P_o(s) = P_o(s')$. Let $a \in A$ and assume that $sa \in \overline{K_1 \parallel K_2}$, $s' \in \overline{K_1 \parallel K_2}$, and $s'a \in L_1 \parallel L_2$. Let $P_i : A^* \rightarrow A_i^*$, for $i = 1, 2$. Then $P_i(sa) \in \overline{K_i}$, $P_i(s') \in \overline{K_i}$, and $P_i(s'a) \in L_i$ imply that $P_i(s'a) \in \overline{K_i}$, by observability of K_i with respect to L_i . Thus, $s'a \in \overline{K_1} \parallel \overline{K_2} = \overline{K_1 \parallel K_2}$. \square

Lemma 16 ([29]). *Let $L_i \subseteq A_i^*$, $i \in J$, and $\bigcup_{k \neq \ell, k, \ell \in J} (A_k \cap A_\ell) \subseteq A_0$. If $P_{i,0} : A_i^* \rightarrow (A_i \cap A_0)^*$ is an L_i -observer, for $i \in J$, then $\|\overline{\bigcap_{i \in J} L_i}\| = \|\bigcap_{i \in J} \overline{L_i}\|$ if and only if $\|\bigcap_{i \in J} P_{i,0}(L_i)\| = \|\bigcap_{i \in J} \overline{P_{i,0}(L_i)}\|$.*

Lemma 17 ([52]). *Let $P_k : A^* \rightarrow A_k^*$ be a projection, and let $L_i \subseteq A_i^*$, where $A_i \subseteq A$, for $i = 1, 2$, and $A_1 \cap A_2 \subseteq A_k$. Then $P_k(L_1 \parallel L_2) = P_k(L_1) \parallel P_k(L_2)$.*

Lemma 18 ([16]). *Let $K \subseteq L \subseteq M$ be such that K is controllable with respect to \overline{L} and L is controllable with respect to \overline{M} . Then K is controllable with respect to \overline{M} .*

Acknowledgements

The authors are grateful to the anonymous reviewers, whose comments and suggestions significantly improved the paper. The work was supported by RVO 67985840, by GAČR in project GA15-02532S and by the German Research Foundation (DFG) in Emmy Noether grant KR 4381/1-1 (DIAMOND).

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